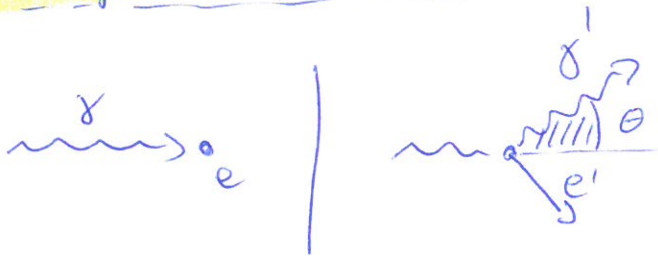
Aufgabenblatt 7 - PEP 3

(1) Compton-Wellenlänge

(a) Compton-Effekt



$$\boxed{\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta) = \lambda_c(1-\cos\theta)}$$

(λ_c = Compton-Wellenlänge $= 2.4$ pm.)

In Energieeinheiten:

$$\lambda' = \lambda_c(1-\cos\theta) + \lambda$$

$$\frac{hc}{E_\gamma'} = \lambda_c(1-\cos\theta) + \frac{hc}{E_\gamma} \quad \Big|\ (\,)^{-1}$$

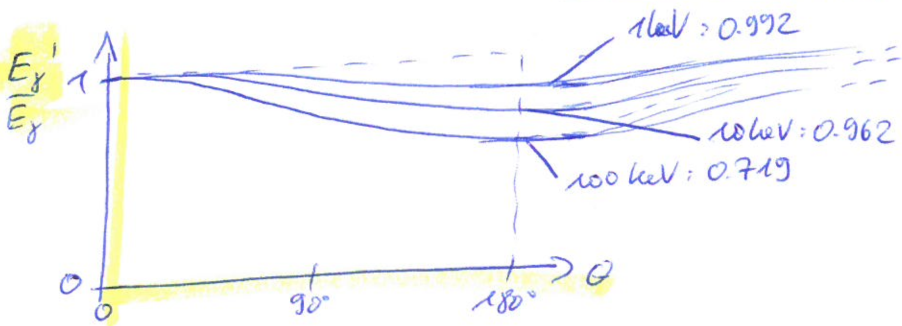
$$\Leftrightarrow \frac{E_\gamma'}{hc} = \frac{1}{\lambda_c(1-\cos\theta) + \frac{hc}{E_\gamma}} \quad \Big|\cdot \frac{hc}{E_\gamma}$$

$$\Leftrightarrow \boxed{\frac{E_\gamma'}{E_\gamma}} = \frac{hc}{E_\gamma\lambda_c(1-\cos\theta) + hc} = \frac{1}{\frac{E_\gamma\lambda_c}{hc}(1-\cos\theta)+1} = \boxed{\frac{1}{1+\frac{E_\gamma}{m_e c^2}(1-\cos\theta)}}$$

mit $m_e c^2 = 511$ keV

$$\boxed{\left(\frac{E_\gamma'}{E_\gamma}\right)_{min} = \frac{1}{1+\frac{E_\gamma}{mc^2}\cdot 2}}$$

höchster E-Übertrag bei $\theta = 180°$

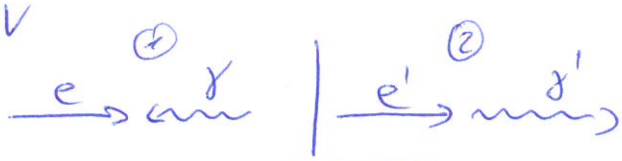


(b) Inversen Compton-Effekt:

(tritt in Umgebung von HE Quellen im Weltraum auf!)

$E_\gamma = 1$ eV (sichtbar)
$E_\gamma' = 1$ GeV

bei $\underline{\theta = \pi}$



relativistisch:

$$\boxed{\begin{aligned} E &= \sqrt{p^2c^2+m^2c^4} \\ \rightarrow p &= \sqrt{\frac{E^2}{c^2} - m^2c^2} \end{aligned}}$$

E-erhaltung: 1):
$$E_e + E_\gamma = E_e' + E_\gamma'$$
$$\hookrightarrow \underline{E_e' = E_e - \Delta E_\gamma} \quad (\Delta E_\gamma = E_\gamma' - E_\gamma > 0!)$$

Impulserhaltung: 2)
$$p_e - p_\gamma = p_e' + p_\gamma'$$
$$\Leftrightarrow \sqrt{\frac{E_e^2}{c^2} - m^2c^2} - \frac{E_\gamma}{c} = \sqrt{\frac{(E_e-\Delta E_\gamma)^2}{c^2} - m^2c^2} + \frac{E_\gamma'}{c}$$

mach...

$$\Rightarrow \dots \Leftrightarrow \underline{E_e} = \frac{1}{2}\left(\Delta E_\gamma + \sqrt{(E_\gamma + E_\gamma')^2 + m^2c^4\left(1 + {}^{E_\gamma}/_{E_\gamma'} + {}^{E_\gamma'}/_{E_\gamma}\right)}\right)$$

$$\approx \frac{1}{2}\left(1\,\text{GeV} + \sqrt{(1\,\text{GeV})^2 + (0.5\,\text{MeV})^2\cdot(1+0+10^9)}\right) \simeq 8\,\text{GeV}$$

(2) Sunjajew – Seldowitsch Effekt

e, m, $v = \beta \cdot c$

Dopplereffekt: $\frac{\Delta\nu}{\nu} = \frac{v}{c}$

$$\rightarrow \boxed{\frac{\nu'}{\nu} = \frac{\nu \pm \Delta\nu}{\nu} = 1 \pm \frac{v}{c} = 1 \pm \beta}$$

Wegen Verschiebungsgesetz: $\nu \sim T$

(a) Strahlungsdruck: $\boxed{p = \sigma_{sb} T^4 \frac{1}{3c}}$

Bremskraft:

p_{rot} (2.7K) → e (m) β ← p_{Blau} (2.7K)

W.Q.

$$m \frac{dv}{dt} = mc \frac{d\beta}{dt} = -\sigma \underbrace{(p_{Blau} - p_{rot})}_{\Delta p}$$

$$\left(\begin{array}{c} \text{Taylor:} \\ (1+\beta)^4 \simeq 1 + 4\beta + \dots \end{array} \right)$$

$$\Delta p = \frac{\sigma_{sb}}{3c} \left[(T(1+\beta))^4 - (T(1-\beta))^4 \right]$$

$$\simeq \frac{\sigma_{sb}}{3c} T^4 \left[1 + 4\beta - 1 + 4\beta \right] = \frac{\sigma_{sb}}{3c} T^4 8\beta$$

$$\Rightarrow mc \frac{d\beta}{dt} = -\sigma \frac{\sigma_{sb}}{3c} T^4 8\beta$$

$$\Leftrightarrow \frac{d\beta}{\beta} = -\sigma \frac{8}{3} \frac{\sigma_{sb}}{mc^2} T^4 dt \quad \Big| \int \qquad \beta(0) = \beta_0, \; \beta(t) = \beta$$

Kraft $\sim \beta$
→ Zerfallsgesetz: Halbwertszeit unab. von β_0

$$\Leftrightarrow \int_{\beta_0}^{\beta} \frac{1}{\beta} d\beta = -\int_0^t \sigma \frac{8}{3} \frac{\sigma_{sb}}{mc^2} T^4 dt$$

$$\Leftrightarrow \ln\beta - \ln\beta_0 = -\sigma \frac{8}{3} \frac{\sigma_{sb}}{mc^2} T^4 t \quad \Big| \exp$$

$$\boxed{\beta(t) = \beta_0 \exp\left(-\sigma \frac{8}{3} \frac{\sigma_{sb}}{mc^2} T^4 t\right)}$$

(b) Abbremszeit? $\beta(t) < \frac{1}{e}\beta_0 \Leftrightarrow +\sigma \frac{8}{3} \frac{\sigma_{sb}}{mc^2} T^4 t > 1$

$$\boxed{t > \frac{3}{8} \frac{mc^2}{\sigma \sigma_{sb} T^4}} \quad \text{qed.}$$

(c) mit $\sigma_{sb} = 5.67 \cdot 10^{-8} \frac{W}{m^2 K^4}$, $\sigma = \sigma_{Thomson} = \frac{8}{3}\pi r_e^2 = 6.6 \cdot 10^{-29}\, m^2$, $T = 2.7K$

$$\rightarrow t_{stop} \approx 1.5 \cdot 10^{20}\, s = 4.7 \cdot 10^{12}\, a \sim 3000 \frac{1}{H_0}$$

Effekt bewirkt rot/blau verschiebung der 2.7K Strahlung "vor" weil
entfernten Galaxienhaufen ---→ Planck-Satellit ..

(3) Dispersion eines Wellenpakets

(a) $E = \hbar\omega \; ; \; E = \frac{p^2}{2m} \rightarrow \boxed{\omega(k) = \frac{\hbar k^2}{2m}}$

(b) Gauß-Wellenpaket:

$$\psi_0(x) = \psi(x,t=0) = A\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$\psi(x,t)$? Ansatz: $\psi(x,t) = \int_{-\infty}^{\infty} dk\, \varphi(k)\, \underbrace{e^{ikx - i\omega t}}_{\text{eb. Welle}}$ $\qquad \omega = \frac{\hbar k^2}{2m}$

$\hookrightarrow \varphi(k)$ AB: $\psi(x,t=0) = \psi_0(x) = \int_{-\infty}^{\infty} \varphi(k)\, e^{ikx}$

$$\Rightarrow \varphi(k) = \int_{-\infty}^{\infty} dx\, \psi_0(x) e^{-ikx} = A\int_{-\infty}^{\infty} dx\, \exp\left(-\frac{x^2}{2\sigma^2} - ikx\right)$$

Erweitere: $\varphi(k) = A\int_{-\infty}^{\infty} dx\, \exp\left(-\left(\frac{x^2}{2\sigma^2} + ikx - \frac{k^2\sigma^2}{2}\right)\right)\exp\left(-\frac{k^2\sigma^2}{2}\right)$

$$= A\int_{-\infty}^{\infty} \exp\Big[-\underbrace{\left(\frac{x}{\sqrt{2}\sigma} + \frac{ik\sigma}{\sqrt{2}}\right)}_{=\frac{x + ik\sigma^2}{\sqrt{2}\sigma} \overset{!}{=} \frac{y}{\sqrt{2}\sigma}}{}^2\Big]\exp\left(-\frac{k^2\sigma^2}{2}\right) dx \qquad (y = x + ik\sigma^2)$$

$$= A\exp\left(-\frac{k^2\sigma^2}{2}\right)\int_{-\infty}^{\infty} dy\, e^{-\frac{y^2}{2\sigma^2}} = 2\pi\sigma A\exp\left(-\frac{k^2\sigma^2}{2}\right)$$

(c) $\psi(x,t) = \int_{-\infty}^{\infty} dk\, \varphi(k) e^{ikx - i\omega t} = 2\pi\sigma A\int_{-\infty}^{\infty} dk\, \exp\left(-\frac{k^2\sigma^2}{2} + ikx - i\omega t\right)$

$$= 2\pi\sigma A\int_{-\infty}^{\infty} \exp\left(-\frac{k^2\sigma^2}{2} - i\frac{\hbar k^2}{2m}t + ikx\right) = 2\pi\sigma A\int_{-\infty}^{\infty} dk\, \exp\Big(-\frac{k^2}{2}\underbrace{\left(\sigma^2 + i\frac{\hbar t}{m}\right)}_{b(t)} + ikx\Big)$$

$$= 2\pi\sigma A\int_{-\infty}^{\infty} dk\, \exp\left[-\left(\frac{k^2}{2}b(t) - ikx\right)\right]$$

Erweitere: $= 2\pi\sigma A\int_{-\infty}^{\infty} dk\, \exp\left[-\left(\frac{k^2 b(t)}{2} - ikx + \frac{x^2}{2b(t)}\right)\right]\exp\left(-\frac{x^2}{2b(t)}\right)$

$$= 2\pi\sigma A\exp\left(-\frac{x^2}{2b(t)}\right)\int_{-\infty}^{\infty} dk\, \exp\Big(-\underbrace{\left(\frac{k\sqrt{b}}{\sqrt{2}} - \frac{ix}{\sqrt{2b}}\right)}_{=: \frac{y}{\sqrt{2b}} = \frac{kb - ix}{\sqrt{2b}}}{}^2\Big) \qquad (y = kb - ix)$$

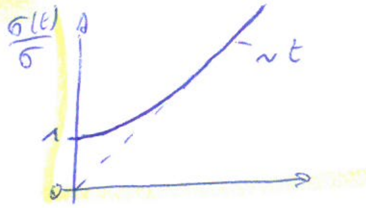
$$= 2\pi\sigma A\exp\left(-\frac{x^2}{2b(t)}\right)\underbrace{\int_{-\infty}^{\infty} dy\, \exp\left(-\frac{y}{\sqrt{2b}}\right)^2}_{2\pi\sqrt{b}}$$

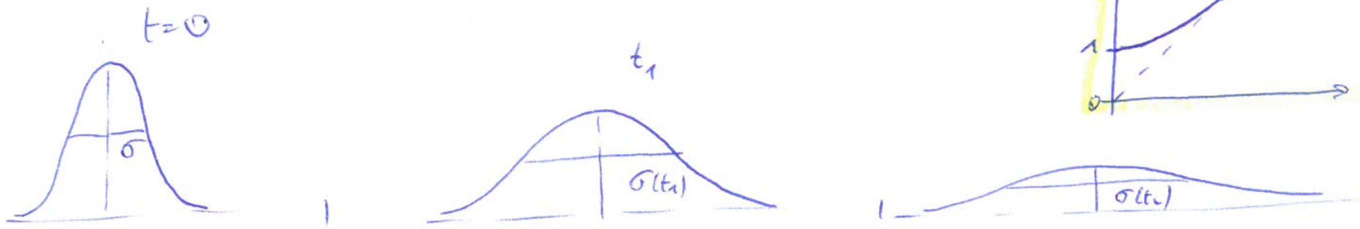
$$\boxed{\psi(x,t) = 4\pi^2\frac{\sigma}{\sqrt{b}} A\exp\left(-\frac{x^2}{2b(t)}\right) \; ; \quad b(t) = \sigma^2\left(1 + i\frac{\hbar}{m\sigma^2}t\right)}$$

(d) $|\psi(x,t)|^2 = \psi(x,t)\cdot\psi^*(x,t) = 16\pi^4|A|^2\cancel{\sigma^2}\frac{1}{\cancel{\sigma}\sqrt{1 + i\frac{\hbar}{m\sigma^2}t}}\frac{1}{\cancel{\sigma}\sqrt{1 - i\frac{\hbar}{m\sigma^2}t}}\exp\left(-\frac{x^2}{2(\sigma^2 + i\frac{\hbar}{m}t)} - \frac{x^2}{2(\sigma^2 - i\frac{\hbar}{m}t)}\right)$

$$= 16\pi^4|A|^2\frac{1}{\sqrt{1 + \frac{\hbar}{m\sigma^2}t}}\exp\left(-\frac{x^2}{\sigma^2\left(1 + \left(\frac{\hbar}{m\sigma^2}\right)^2 t^2\right)}\right)$$

(e) Breite = $\boxed{\sigma(t) = \sigma\sqrt{1+\left(\frac{\hbar}{m\sigma^2}\right)^2 t^2}}$





(f) $m = 1g$, $\sigma = 0.1mm$ $\quad \sigma(t) = 0.14\,mm$

$$\sigma(t) = \sigma\sqrt{1+\left(\frac{\hbar}{m\sigma^2}\right)^2 t^2} \quad \Big|^2$$

$$\sigma(t)^2 = \sigma^2\left(1+\left(\frac{\hbar}{m\sigma^2}\right)^2 t^2\right)$$

$$t^2 = \left[\frac{\sigma(t)^2}{\sigma^2} - 1\right]\left(\frac{m\sigma^2}{\hbar}\right)^2$$

$$\underline{\underline{t}} = \sqrt{\left(\frac{\sigma(t)}{\sigma}\right)^2 - 1}\left(\frac{m\sigma^2}{\hbar}\right) = \sqrt{\left(\frac{0.14}{0.1}\right)^2 - 1}\left(\frac{0.001\,kg \; 1\cdot 10^{-8} m^2 \; s}{1.055\cdot 10^{-34}\,kg\,m^2}\right) \triangleq 10^{23}\,s$$

$$= \underline{\underline{3\cdot 10^{15}\,a}} \quad (!)$$

(g) $\sigma(t) < \sigma$ (Rückgängige Dispersion)

$$\Leftrightarrow \left(\frac{\hbar}{m\sigma^2}\right)^2 < 0 \qquad \text{Masse} \in \mathbb{C}^\alpha \quad (i\mathbb{R})$$

Tachyonen: hypothetische Teilchen mit $v > c$
damit ihre Energie positiv ist, muss wegen
$E = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}$ $m_0 \in \mathbb{C}^R$ sein

(h) <u>Python</u>.